An Optimal Mastermind Strategy

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1 Introduction

Mastermind[1] is a game for two players, a codemaker and a codebreaker. The codemaker chooses a secret code consisting of four pegs with six possible colors. Repeated colors are allowed, so the set of possible codes is $6^4 = 1296$. The codebreaker repeatedly guesses a code and receives a hint from the codemaker until the codebreaker obtains the secret code exactly. The goal of the codebreaker is to minimize the number of guesses needed.

More precisely, after the codebreaker guesses $\overline{x} = x_1x_2x_3x_4$, if the secret code is $\overline{y} = y_1y_2y_3y_4$, the codemaker gives the hint $h(\overline{x}, \overline{y}) = (b(\overline{x}, \overline{y}), w(\overline{x}, \overline{y}))$, which is defined as follows. We denote the colors by 1, 2, 3, 4, 5, 6.

1. The number $b(\overline{x}, \overline{y})$ of "black hits" is defined as $b(\overline{x}, \overline{y}) = \#\{j : x_j = y_j\}$.
2. The number $w(\overline{x}, \overline{y})$ of "white hits" is defined as $w(\overline{x}, \overline{y}) = \sum_{i=1}^{6} \min(m_i, n_i) - b(\overline{x}, \overline{y})$, where $m_i = \#\{j : x_j = i\}$ and $n_i = \#\{j : y_j = i\}$.

Thus, $b(\overline{x}, \overline{y})$ is the number of pegs in the guess that are correct colors in correct positions, and $b(\overline{x}, \overline{y}) + w(\overline{x}, \overline{y})$ is the number of pegs that are correct colors in either correct or incorrect positions. Note that there are 14 possible hints.

In the first substantive analysis of Mastermind, Knuth [1] demonstrated a strategy that requires at most five guesses in the worst case. If each secret code is equally probable, the expected number of guesses with Knuth's strategy is 4.478. Irving [2] later demonstrated a strategy that requires only 4.369 guesses in the expected case. Neuwirth [3] improved this bound to 4.364. However, no optimal strategy has been demonstrated. More recently, Chvátal [4], Flood [5, 6, 7], Ko and Teng [8], and Koyama and Lai [9, 10] have investigated some generalizations and variants of the Mastermind game.

In this note, we demonstrate an optimal strategy for Mastermind. Our strategy requires at most six guesses in the worst case and $5625/1296 \approx 4.340$ in expected case. We also show how to modify the strategy to use at most five guesses in the worst case; this increases the

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[1] *Mastermind* is a registered trademark of Invicta Plastics (USA) Ltd.

expected number of guesses to $5626/1296 \approx 4.341$. Our strategy was obtained by an exhaustive computer search.

In the next section, we describe some optimizations used for obtaining an optimal strategy. We describe our strategy in Section 3.

2 Finding an optimal strategy

To obtain an optimal Mastermind strategy, we performed an exhaustive depth-first computer search. To reduce the search space, we employed a variety of optimizations; we discuss only the most important ones below.

The first optimization relies on the notion of *equivalence*, which was used by Neuwirth [3]. We define an *equivalence transformation* (e.t.) to be a composition of a permutation on the set of positions and a permutation on the set of colors. For example, if t is an equivalence transformation defined by the permutation $\binom{1\,2\,3\,4\,5\,6}{3\,6\,1\,5\,2\,4}$ on the set of colors and the permutation $\binom{1\,2\,3\,4}{1\,4\,3\,2}$ on the set of positions, then $t(1112) = 3633$. An important property of the Mastermind hint function is that, for any e.t. t and any codes $\overline{x}$ and $\overline{y}$, we have $h(\overline{x}, \overline{y}) = h(t(\overline{x}), t(\overline{y}))$.

We can reduce the search space with the following observation. Suppose we have previously made guesses $\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_{k-1}$. We say that two codes $\overline{c}$ and $\overline{d}$ are *case equivalent* if there exists an e.t. t such that $t(\langle \overline{g}_1, \ldots, \overline{g}_{k-1}, \overline{c} \rangle) = \langle \overline{g}_1, \ldots, \overline{g}_{k-1}, \overline{d} \rangle$. It is possible to show that if some codes $\overline{c}_1, \ldots, \overline{c}_n$ are case equivalent, then we need to consider only one code $\overline{c}_1$ as a kth guess for computing an optimal strategy.

For the first guess, there are only five nonequivalent guesses: 1111, 1112, 1122, 1123, and 1234. It is straightforward to show that the guess 1111 yields a suboptimal strategy, so we need to consider only four initial guesses.

During guesses 2 and 3, we test codes for case equivalence by computing a canonical integer for each code sequence as follows. We apply every possible position-permutation to the code sequence, and, for each permutation, we permute the colors canonically using a greedy algorithm and encode the resulting sequence as a 32-bit integer. We then choose the lowest such integer as our canonical integer encoding.

Other optimization we use rely on the notion of *candidates*. Let S be the set of all codes. We define the *current set of candidates* to be the subset of codes that are consistent with previous guesses and hints. More precisely, if the codebreaker has made guesses $\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_k$ and received hints $(b_1, w_1), (b_2, w_2), \ldots, (b_k, w_k)$, then the set C of candidates is

$$C = \{\overline{x} \in S : b(\overline{g}_i, \overline{x}) = b_i \text{ and } w(\overline{g}_i, \overline{x}) = w_i\}.$$

Note that the actual secret code must be a candidate.

Clearly, we do not have to consider guesses that do not reduce the number of candidates, and we can trivially compute an optimal strategy when the current number of candidates is at most 2. Also, when the number of candidates is small (less than 14), we try to limit the amount of recursion used by first testing if there are any obviously optimal guesses. For example, we test if there is any candidate $\overline{c}$ such that each reply to a guess of $\overline{c}$ uniquely determines the secret code; if not, then for each inconsistent code $\overline{d}$, we test if each reply to $\overline{d}$ uniquely determines the secret code.

3 The strategy

Below, we describe our optimal Mastermind strategy. Our strategy uses $5625/1296 \approx 4.340$ guesses in the expected case and at most 6 guesses in the worst case. The best strategy known previously is due to Neuwirth [3]. Neuwirth's strategy uses $5656/1296 \approx 4.364$ guesses in the expected case and at most 6 guesses in the worst case.

Our strategy was calculated using an HP 9000 Series 720 computer.

To describe our strategy, we use the notation of Irving [2]. If there are n candidates remaining at some given stage, the situation is represented by

n if $n \leq 1$;

$n(y_1y_2y_3y_4)$ if $n \geq 2$, and the reply to $y_1y_2y_3y_4$ will uniquely determine the secret code, and where the presence of * denotes that $y_1y_2y_3y_4$ is not a candidate;

$n(y_1y_2y_3y_4 : \alpha_{04}, \alpha_{03}, \alpha_{02}, \alpha_{01}, \alpha_{00};\ \alpha_{13}, \alpha_{12}, \alpha_{11}, \alpha_{10};\ \alpha_{22}, \alpha_{21}, \alpha_{20};\ \alpha_{30};\ \alpha_{40})$
if $n > 2$, and the guess $y_1y_2y_3y_4$ and reply (i, j) lead to situation α_{ij}.

The strategy is shown in Table 1.

Our strategy differs substantially from Neuwirth's in parts A, B, E, G, H, and I, and differs slightly in parts C, D, and F.

Note that our strategy may use six guesses only in the case when we receive a hint $(1, 1)$ after an initial guess of 1123 and the hint $(1, 1)$ after a second guess of 1415. By handling this situation with the modification below, we obtain a strategy that is optimal among all strategies that use at most five guesses in the worst case. The expected number of guesses for this modified strategy is $5626/1296 \approx 4.341$. Note that any Mastermind strategy requires at least five guesses in the worst case.

K = 39 (1256: 1, 5 (6135), 10 (4335*), 1, 0; 1, 2 (1642), 6 (1161*), 2 (1334); 1, 3 (1536), 3 (1242); 3 (1112*); 1)

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Table 1. An optimal Mastermind strategy.

1296 (1123: 2 (2311), 44A, 222B, 276C, 81D; 4(1312), 84E, 230F, 182G; 5 (1321: 1, 0, 0, 0, 0; 2(1132), 0, 0, 0; 1, 0, 0; 0; 1), 40H, 105I; 20J; 1)

A = (2345: 0, 6 (3412), 10 (3612: 1, 0, 0, 0, 0; 1, 1, 2 (4211), 0; 1, 0, 2 (3411); 1; 1), 2 (3611), 0; 0, 6 (3215), 8 (2461: 0, 1, 1, 0, 0; 0, 0, 1, 1; 0, 1, 2 (2231); 1; 0), 4 (2611); 0, 2 (2314), 4 (2316); 2 (2315); 0)

B = (2434: 1, 12 (3245: 1, 1, 0, 0, 0; 1, 1, 1, 0; 1, 2 (3342), 1; 2 (3242); 1), 34 (3541: 1, 2 (4215), 6 (5362), 1, 0; 1, 3 (4251), 7 (3265), 4 (3266); 1, 2 (5241), 3 (3552); 2 (3341); 1), 36 (3551: 1, 3 (5316), 5 (5612), 3 (6216), 0; 0, 4 (3315), 5 (3316), 2 (4611); 2 (3515), 3 (3516), 4 (3361); 3 (3351); 1), 4 (1515*); 2 (3244), 17 (4235: 1, 3 (2342), 1, 0, 0; 2 (2345), 1, 2 (4314), 0; 1, 2 (4332), 1; 2 (4232); 1), 44 (3235: 0, 2 (2352), 5 (5412), 3 (2641), 1; 1, 6 (5632), 7 (3314), 6 (3114*); 1, 4 (3332), 3 (6232); 4 (3236); 1), 26 (2516: 0, 1, 4 (3631), 1, 0; 1, 4 (2251), 4 (3531), 0; 2 (2561), 3 (2215), 1; 4 (2216); 1); 3 (2344), 11 (2354: 1, 2 (3432), 1, 0, 0; 1, 2 (3234), 2 (2441), 0; 0, 0, 0; 1; 1), 23 (2325: 0, 0, 1, 2 (3431), 0; 1, 4 (2536), 5 (2214), 3 (2461); 1, 3 (2535), 2 (2336); 1; 0); 8 (2324: 0, 0, 0, 0, 0; 1, 2 (2435), 0, 0; 1, 2 (2534), 1; 1; 0); 1)

C = (2445: 1, 12 (4562: 1, 1, 0, 0, 0; 2 (4256), 0, 1, 0; 1, 2 (4252), 2 (4514); 1; 1), 42 (5266: 1, 1, 5 (3654), 6 (3534), 1; 1, 3 (4662), 8 (4516), 5 (5354); 2 (5662), 2 (5562), 3 (4262); 3 (5256); 1), 38 (3364: 0, 1, 4 (4616), 6 (5616), 1; 1, 3 (3536), 6 (6356), 6 (6266); 2 (3634), 2 (3356), 2 (3566); 3 (3664); 1), 9 (2363: 0, 1, 1, 1, 0; 0, 2 (3336), 1, 1; 0, 1, 1; 0; 0); 3 (4452), 27 (2564: 3 (5246), 5 (4255), 2 (4242), 0, 0; 3 (4265), 3 (4462), 6 (3454), 0; 1, 1, 1; 1; 1), 51 (2566: 0, 2 (6255), 6 (4615), 8 (6434), 5 (3344); 1, 4 (2652), 8 (3456), 8 (4416); 1, 2 (2256), 3 (2264); 2 (2556); 1), 36 (3646: 0, 2 (6365), 4 (5365), 5 (5355), 1; 1, 2 (6416), 6 (5635), 5 (1353*); 2 (3466), 3 (3436), 3 (3635); 1; 1); 4 (2454), 15 (2456: 1, 1, 1, 0, 0; 0, 3 (2644), 5 (4415), 0; 1, 1, 0; 1; 1), 28 (3456: 1, 1, 4 (2642), 1, 0; 2 (3645), 4 (3345), 6 (2646), 3 (2255); 1, 1, 2 (2466); 2 (3446); 0); 9 (1246: 0, 1, 1, 0, 0; 0, 1, 3 (2442), 1; 0, 1, 1; 0; 0); 1)

D = (4456: 2 (5644), 8 (5645: 0, 1, 0, 0, 0; 1, 0, 0, 0; 1, 2 (5544), 1; 1; 1), 6 (4555*), 0, 0; 4 (4564), 18 (4565: 1, 2 (5646), 2 (5444), 0, 0; 2 (5546), 2 (5445), 2 (4644), 0; 1, 1, 2 (4544); 2 (4545); 1), 8 (5655), 2 (5555); 5 (4546: 1, 0, 0, 0, 0; 2 (4465), 0, 0, 0; 1, 0, 0; 0; 1), 10 (4565: 0, 1, 1, 0, 0; 1, 1, 1, 0; 1, 1, 2 (4445); 1; 0), 9 (5615: 0, 1, 0, 1, 1; 0, 1, 2 (4555), 1; 0, 1, 1; 0; 0); 8 (4545: 0, 1, 1, 0, 0; 0, 1, 2 (4466), 0; 1, 1, 1; 0; 0); 1)

E = (1415: 0, 6 (3141: 0, 0, 0, 0, 0; 0, 0, 0, 0; 1, 1, 1; 2 (2141); 1), 12 (3142: 1, 1, 0, 0, 0; 1, 1, 2 (2161), 0; 1, 1, 2 (3131); 1; 1), 10 (2136: 2 (3621), 2 (3221), 0, 0, 0; 1, 1, 0, 0; 1, 1, 0; 1; 1), 0; 0, 9 (1251: 1, 2 (2114), 1, 0, 0; 0, 0, 0, 0; 0, 1, 1; 2 (1241); 1), 19 (1631: 1, 2 (2116), 4 (2513), 0, 0; 0, 1, 5 (1342), 0; 1, 1, 2 (1234); 1; 1), 10 (1236: 1, 1, 0, 0, 0; 2 (1362), 2 (2213), 0, 0; 1, 1, 0; 1; 1); 0, 6 (1314), 9 (1261: 0, 0, 1, 0, 0; 1, 1, 1, 0; 1, 1, 2 (1235); 1; 0); 3 (1215); 0)

F = (1415: 2 (4151), 12 (2154: 1, 0, 0, 0, 0; 1, 0, 4 (5161), 0; 1, 1, 2 (5151); 1; 1), 34 (6152: 1, 4 (2521), 5 (2543), 0, 0; 1, 3 (2146), 5 (3164), 3 (3144); 2 (2156), 2 (3156), 4 (2142);

3 (2152); 1), 41 (4233: 1, 3 (2324), 4 (3522), 6 (2621), 0; 1, 6 (3224), 4 (2563), 3 (2221); 1, 4 (2533), 4 (5263); 3 (4243); 1), 20 (2663: 1, 3 (3226), 2 (3222), 0, 0; 1, 3 (2326), 2 (2322), 0; 1, 1, 2 (2233); 3 (2263); 1); 2 (1541), 19 (1561: 1, 2 (4116), 4 (2145), 0, 0; 0, 1, 5 (1254), 0; 1, 1, 2 (1534); 1; 1), 39 (3155: 1, 4 (1536), 8 (1346), 9 (1264: 1, 2 (2421), 0, 0, 0; 1, 0, 0, 0; 1, 1, 1; 1; 1), 0; 0, 3 (1356), 5 (1256), 2 (6111); 1, 1, 1; 3 (3135); 1), 23 (3613: 1, 1, 3 (1262), 2 (2325), 0; 0, 1, 5 (2433), 5 (3225); 1, 0, 1; 2 (3313); 1); 3 (1451), 11 (1461: 0, 1, 1, 0, 0; 1, 1, 3 (1245), 0; 0, 0, 2 (1452); 1; 1), 17 (1365: 0, 1, 2 (3413), 0, 0; 0, 1, 4 (1611), 1; 1, 1, 2 (1255); 3 (1335); 1); 6 (1461*); 1)

G = (1445: 1, 6 (1156*), 19 (1656: 0, 1, 2 (4563), 5 (4522), 1; 1, 1, 2 (4526), 3 (4353); 1, 1, 1; 0; 0), 42 (3556: 0, 3 (5363), 5 (4363), 6 (4622), 2 (2224); 2 (5563), 6 (5353), 4 (5226), 4 (2522); 1, 4 (3563), 4 (2526); 1; 0), 16 (2626: 0, 0, 1, 2 (3363), 1; 0, 1, 3 (3663), 1; 2 (6226), 0, 1; 3 (2226); 1); 1, 12 (4165: 1, 1, 1, 0, 0; 2 (1564), 1, 2 (4453), 0; 0, 1, 1; 1; 1), 33 (1566: 0, 1, 4 (4625), 8 (5343), 5 (2424); 1, 2 (5165), 3 (5426), 5 (4525); 1, 1, 0; 1; 1), 22 (2562: 1, 3 (5225), 1, 4 (3643), 2 (3343); 1, 4 (2426), 1, 3 (1666); 0, 1, 1; 0; 0); 3 (1454), 8 (4416*), 12 (5165: 0, 0, 1, 1, 2 (3443); 1, 1, 2 (1466), 1; 1, 1, 1; 0; 0); 6 (4456*); 1)

H = (1245: 0, 4 (3124), 8 (3126: 0, 0, 0, 0, 0; 1, 1, 2 (4113), 0; 0, 1, 1; 1; 1), 2 (3113), 0; 0, 6 (1324), 8 (2166: 0, 1, 1, 2 (1413), 0; 0, 1, 1, 1; 0, 1, 0; 0; 0), 4 (1136); 0, 3 (1253), 4 (1263); 1; 0)

I = (1245: 1, 10 (4162: 0, 0, 1, 0, 0; 1, 2 (2124), 1, 0; 1, 1, 2 (4153); 1; 0), 23 (3426: 0, 1, 3 (5163), 2 (5133), 0; 1, 3 (5623), 4 (2523), 2 (2122); 1, 2 (2423), 3 (2126); 1; 0), 10 (3166: 0, 1, 2 (2623), 1, 0; 0, 1, 1, 1; 1, 0, 1; 1; 0), 0; 2 (1524), 12 (1436: 0, 1, 1, 2 (2125), 0; 0, 1, 1, 1; 0, 1, 3 (1422); 1; 0), 18 (1356: 0, 1, 3 (3143), 1, 0; 1, 2 (1463), 3 (1433), 1; 1, 1, 2 (1151); 2 (1156); 0), 11 (1161: 0, 0, 0, 1, 2 (2223); 0, 0, 1, 1; 1, 0, 2 (1363); 2 (1111); 1); 1, 4 (1625), 11 (3164: 1, 2 (1343), 1, 1, 0; 0, 1, 1, 2 (1115); 0, 0, 2 (1144); 0; 0); 2 (1145); 0)

J = (1245: 0, 2 (4123), 3 (2116*), 0, 0; 0, 3 (1124), 6 (1162*), 3 (1116*); 0, 1, 2 (1143); 0; 0)